

Original Form of Kepler’s Third Law and its Misapplication in Propositions XXXII-XXXVII in Newton’s *Principia* (Book I)

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Abstract: The original form of Kepler’s Third Law contains a caveat regarding the requirement of small eccentricities – a fact that has been missed by the traditional Newtonian derivations. This constraint is analyzed, and a re-clarification of the real meaning of “mean distance” in the law is provided, by following up the indications given by Kepler in the *Harmonices Mundi*. It is shown that the modified expression for the “mean distance” not only clears up conceptual difficulties, but also removes a discrepancy found by Kepler for Mercury. Based on this re-evaluation, the result of ignoring the small-eccentricity constraint is analyzed in the Propositions XXXII-XXXVII in the *Principia*. It is seen that there are several conceptual and mathematical mistakes that are inevitable with the Newtonian form of Kepler’s Third Law.

Introduction

Kepler’s Third Law, also called the Harmonic Law, which states the empirical relationship between the mean distances of the planets from the sun to their orbital periods, has been part of the foundation of astronomy for four centuries. Newton’s *Principia* built on this relationship to derive the inverse square law, as is admitted by Newton himself [1]. It is hence of vital interest to thoroughly examine the details of its derivation to determine both the mathematical and physical implications. The most general form of this law is expressed as follows:

$$r^3 \propto T^2 \quad (1)$$

Here T is the time period of the planet in question (or moon, if the revolutions around a planet are concerned) and r is the “mean distance.” The meaning of this “mean distance” has been translated in a variety of ways, as summarized, for example, in [2]. The most common understanding is where it is equated to the semi-major axis of an ellipse, which is from Newton [3], and is reaffirmed by most textbooks in physics [4] [5] [6] [7]. Some derivations even start with the inverse-square law or energy theorems, but since that is opposite to the original mode of derivation, the argument becomes circular, and hence not relevant.

Since there has been some uncertainty about the mean distance as Kepler meant it, there have been repetitive recent attempts to ‘average’ the distance of the orbiting object from the force center in various ways. The average value $\langle r \rangle_p$ with respect to a variable p has been done by

utilizing integration to obtain $\langle r \rangle_p = (1/p) \int r dp$. In the calculations in literature, with a as semi-major axis, b as semi-minor axis and e as the eccentricity, the averages are taken with respect to:

(i) Time, $\langle r \rangle_t = a \left(1 + \frac{e^2}{2}\right)$: [2] [8] [9] [10] [11] [12] [13] [14]

(ii) Arc length, $\langle r \rangle_s = a$: [2] [8] [9] [11] [12] [13] [15]

(iii) Angle, $\langle r \rangle_\theta = b$: [2] [8] [9] [11] [13] [14]

Apart from this, Silva [16] has calculated (i), (ii) and (iii) not only with respect to $\langle r \rangle$ but with 20 other variables. There has also been some discussion whether or not Kepler would hypothetically agree with this evaluation, particularly about $\langle r \rangle_s = a$, the semi-major axis of the ellipse:

Would Kepler feel comfortable with the basic idea of this proof, though calculus had not been available in his time? Probably "yes". [13]

It is not clear, how Kepler calculated the average distance of the planet, but most likely he was averaging over the angle. [2]

In most cases the conclusion reached is that the average used by Kepler is $\langle r \rangle_s = a$. However, the question can, and must, be posed: “What average distance did Kepler *himself* intend to portray in his Harmonic Law?” This does not result from merely a historical or speculative interest, but has a direct consequence in determining the physical meaning of the law, and thereby on the derivations of Newton and his successors. Hence in Section 1, Kepler’s original indications will be analyzed, and shown to be different than what is conventionally assumed. In Section 2, the effect of this difference in formulation on the derivations of Propositions XXXI and XXXII (Book I) in the *Principia* will be described. These facts were already mentioned briefly with regard to Propositions X-XVI in a companion paper [17] along with a demonstration of how Newton utilized the Harmonic Law to derive the inverse square law, but they will be elaborated more in this work.

1. Kepler’s Harmonic Law

The full quote of this discovery in Kepler’s works occurs in *Harmonices Mundi* (Harmony of the World) Book V [18] and is reproduced here along with its translation (italics by Kepler, bold by author):

*Sed res est certissima exactissimaque, quòd proportio quae est inter binorum quorumcunque Planetarum tempora periodica, sit praecisè sesquialtera proportionis mediarum distantiarum, id est Orbium ipsorum; **attento tamen hoc, quòd medium arithmeticum inter utramque diametrum ellipticae Orbitae sit paulò minus longiore diametro.****

*** Nam in Commentarijs Martis cap. XLVIII. fol. 232. probavi Medium hoc arithmeticum vel ipsam. Esse diametrum circuli qui longitudine aequat ellipticam orbitam veleá proximè minus.**

Translation:

But it is absolutely certain and exact that *the ratio which exists between the periodic times of any two planets is precisely the ratio of the 3/2th power of the mean distances, i.e., of the spheres themselves; provided, however, that the arithmetic mean between both diameters of the elliptic orbit be slightly less than the longer diameter.**

***For in the *Commentaries on Mars*, chapter XLVIII, page 232, I have proved that this Arithmetic mean is either the diameter of the circle which is equal in length to the elliptic orbit, or else is very slightly less.**

Here, the first part of the quote is repeated in almost all of the references to the law, but (at least in the knowledge of the author) there has been no systematic analysis, or even a mention, of the second part (in bold) and its corresponding footnote. Quite surprisingly, the one author [19] who has addressed the presence of the latter part and the footnote, states that:

The qualification about the arithmetic mean of the diameters is hard to understand. That the “mean between two diameters... must be little less than the longer diameter” is a roundabout way of saying the two axes of the ellipse must be almost equal – in other words, that the eccentricity must be small. In a side note, Kepler gives a clue to his intentions, repeating from the *Astronomia Nova* his contention that the mean of the two diameters of an ellipse either equals or is slightly less than the diameter of a circle with the same circumference as the ellipse.

Since this is the *only* clue provided by Kepler about the qualifications for his law, it is surprising that there has been no follow-up with it to determine the type of averaging Kepler indicated. The notion of the mean distance has been used throughout the *Harmonices Mundi*, without ever referring it to the two diameters of an ellipse. Hence there must be a reason for its presence in this juncture that is not restricted to applying the condition for small eccentricity in a “roundabout way”.

There are two lines of thought suggested by the above quote:

1. The “mean distance” he is talking about in case of an ellipse is slightly different than the one employed in case of circles (i.e. average of the extreme distances.)
2. The relationship established is dependent on the eccentricity being small.

For the first, it is necessary to understand the sections in *Commentaries on Mars* i.e. *Astronomia Nova* Chapter 48 with reference to *Harmonices Mundi*, and for the second, consequences of the eccentricity constraint have to be studied in the theory of celestial dynamics. These will be pursued in Section 2 and 3 respectively.

2. Kepler’s “mean distance” for an ellipse

An error in the translation of a section of *Harmonices Mundi* [18] in Koyré [20] has also added to considerable confusion in the definition of the mean distance, with several works – [2] [13] for example – taking the mistranslation as the actual definition:

It follows (d) that the planet, when it has covered **a quadrant (missing in Koyré)** of its entire circuit, starting at aphelion, is (e) distant from the Sun by an amount which is exactly [equal] to the [arithmetical] mean of its maximum distance at aphelion and its minimum distance at perihelion.

Without the words “a quadrant”, it makes it appear as if the definition of the average over the complete circuit is given by the semi-major axis. However, only the distance from the Sun at a particular point in the circuit is being described here, and there is hence no relation to the Harmonic Law.

It is only by following the footnote to the Harmonic Law, i.e. Chapter 48 in *Astronomia Nova* can the issue be clarified. The purpose of Chapter 48 is mentioned by Kepler in his summary ([21], pg. 97):

4. Another contrivance for seeking the length of the oval path, which is however, accompanied by other geometrical theories. For two circles are given, and two means between them, one arithmetical and the other geometrical, by the former of which a greater circle is constructed, and by the latter, a lesser. Then, by two arguments, the ellipse is shown to be equal to the arithmetic mean...

Hence the objective is to provide an estimate for the perimeter of an ellipse by a corresponding circle. The relevant diagram ([21], pg. 484), has been reproduced in Fig. 1, modified for clarity.

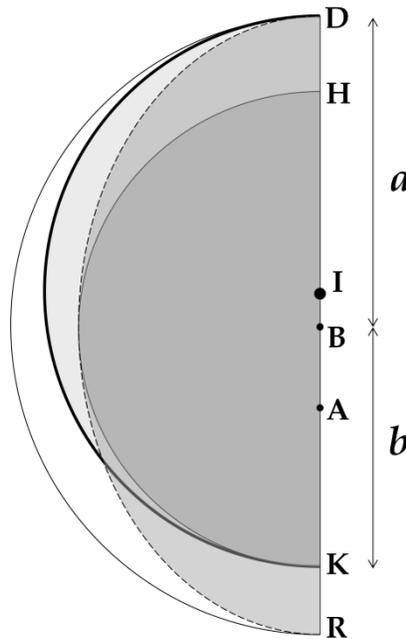


Fig. 1: Circle DK (bold line) centered on I is the “arithmetical mean,” $BD = a$ and $BK = b$, and DR (dashed line) is the ellipse. A is the focus of the orbital movement.

The radius and perimeter of the circle DK in this approximation is:

$$r_{DK} = \frac{a+b}{2} \quad (2a)$$

$$P_{DK} = \pi(a + b) \approx P_{DR}(\text{ellipse}) \quad (2b)$$

By seeking a circle whose circumference is the same as the perimeter of the ellipse P_{DR} , and linking his Harmonic Law to that, Kepler has indicated the need for a different “mean distance” . The mean used in the previous work [17] is from Eq. 2b. Incidentally it may also be noted that Equation 2b does give a close approximation to the actual value of P_{DR} for small eccentricities, per the approximation of Ramanujan [22].

When comparing a body that is moving in an ellipse with semi-major axis a to the same body moving in a circle of radius a , conserved quantities between the two scenarios must be determined. Kepler’s second law states that the product of distance and velocity is a constant, and for two equivalent orbits the value of the time period *must be the same*. In other words, it is necessary to equate their areal velocities (distance*velocity) by keeping the time period constant. Let a be the radius of the circle, P_{ellipse} the perimeter of the ellipse, $\langle r \rangle_{\text{ellipse}}$ the “mean radius” of the ellipse, v the average velocity and T the time period. Then:

$$\begin{aligned} \langle r \rangle_{\text{circle}} * v_{\text{circle}} &= \langle r \rangle_{\text{ellipse}} * v_{\text{ellipse}} \\ \langle r \rangle_{\text{circle}} * \frac{P_{\text{circle}}}{T} &= r_{\text{ellipse}} * \frac{P_{\text{ellipse}}}{T} \\ a * 2\pi a &= \langle r \rangle_{\text{ellipse}} * P_{\text{ellipse}} \end{aligned} \quad (3)$$

Solving this for $\langle r \rangle_{\text{ellipse}}$ by substituting the expression for the perimeter of an ellipse with semi-major axis a :

$$\langle r \rangle_{\text{ellipse}} = a \frac{2\pi a}{P_{\text{ellipse}}} = a \frac{2\pi a}{4a * \int_0^{\pi/2} \sqrt{1-e^2 \sin^2 \theta} d\theta} = \frac{\pi a}{2E(e)} \quad (4)$$

Here, $E(e)$ is the elliptical integral of the second kind, provided by the integral tables [23]. The effect of the eccentricity is seen clearly, as the value of $\langle r \rangle_{\text{ellipse}}$ ranges from a to $a\pi/2$ as e ranges from 0 to 1. Hence, the more a circle gets “squashed” in one direction, for the same time period the corresponding value of “mean distance” increases.

A way to test this result is to compare it with the *only* table of values where Kepler ever utilized the Harmonic Law to derive distances:

Table 1. Derivation of Orbital distances from Harmonic Law

<i>Numbers of the mean movements</i>		<i>Numbers of the ratio of the spheres found among the squares</i>	<i>Semi-diameters as above</i>	<i>Eccentricities in dimensions</i>		<i>Extreme intervals Resulting</i>	
<i>In the original dimensions</i>	<i>In the new dimensions found in inverse order among the cubes</i>			<i>Private as above</i>	<i>Common</i>	<i>Aphelion</i>	<i>Perihelion</i>
156,917	29,539,960	9,556	85	5	562	10,118	8,994
390,263	11,877,483	5,206	85,222	4,222	258	5,464	4,948
2,467,584	1,878,483	1,523	55	5	38	1,661	1,384
4,635,322	1,000,000	1,000	95,178	1,647	17	1,017	983
7,571,328	612,220	721	99,295	705	5	726	716
18,864,680	245,714	392	80,625	17,375	85	476	308

(Reproduced from [24])

As mentioned in *Harmonices Mundi*, it is only in the case of Mercury, whose eccentricity is large, that the mean distance from the Harmonic Law (set as the semi-major axis) overshoots the experimental value of the semi-major axis. Hence, below this table, Kepler concludes:

In Mercury alone is there some small difference. For astronomy is seen to give the following intervals to it: 470, 388, 306, all shorter. It seems that the reason for the dissonance may be referred either to the fewness of the observations or to the magnitude of the eccentricity. (See Chapter III). But I hurry on to the end of the calculation.

This discrepancy is the most likely reason Kepler modified the Harmonic Law to include the caveat regarding small eccentricities. The way in which the caveat was phrased suggests that he intended to use Eq. 2 (or as modified, Eq. 4) in his expression. Applying Equation (4) with $E(e)$ giving 0.98829 for Mercury's eccentricity of 0.215503 (from Table 1, last row, 17,375/80,625) gives the values:

<i>Kepler's calculations:</i>	$a_{max} = 476$	$a = 392$	$a_{min} = 308$
<i>Kepler's data:</i>	$a_{max} = 470$	$a = 388$	$a_{min} = 306$
<i>Corrected using Eq(4):</i>	$a_{max} = 470.4$	$a = 387.4$	$a_{min} = 304.4$

Had Kepler corrected the equation in this way, it would have brought the values that were computed using harmonic intervals closer to the actual data, and shown that eccentricity does

affect the value of the “mean distance” taken. Hence, one can state: *this makes $\langle r \rangle_{\text{ellipse}}$ the most likely candidate for the “mean distance” according to Kepler.*

Comparing $\langle r \rangle_{\text{ellipse}}$ with the average (ii) discussed in the previous section, $\langle r \rangle_s = a$ must be disqualified, since it is not dependent on the eccentricity at all, and therefore one cannot apply the latter part of Kepler’s law to it. However, historically $\langle r \rangle_s = a$ has been the only “mean distance” utilized in celestial dynamics, following the lead of Isaac Newton. The consequences of this assumption will be studied in the next section.

3. Mean distance in the *Principia*

It has been described in Part I of this work ([17]) that in the *Principia*:

(I) In Proposition X, an error of ignoring the area constant was corrected, whereby the time periods of ellipses on the same major axis but different minor axes are seen to be proportional to the minor axes, and not equal as claimed.

(II) In Propositions XI-XIV, by *assuming* the inverse square law to hold between different orbiting bodies, Kepler’s Law was *also* implicitly assumed with the “mean distance” being set *equal* to the semi-major axis a . In Proposition XV this Law is made explicit once more as $a^3 \propto T^2$.

In view of the discussion in Section 2, it must be emphasized again that the following statement from the *Principia* (Book I, Proposition XV, corollary):

Therefore the periodic times in ellipses are the same as in circles whose diameters are equal to the major axes of the ellipses.

And the following portion of Kepler’s Harmonic Law:

... provided, however, that the arithmetic mean between both diameters of the elliptic orbit be slightly less than the longer diameter.

Are in direct contradiction. Since Kepler provides the caveat, the burden of proving the corollary to Proposition XV falls on Newton, and since he begins by *assuming* it implicitly in Proposition XI-XIV, it is clear that no proof exists, mathematically. Also, since Kepler based his law solely on empirical observations, if one looks for empirical evidence to back Newton’s claim that $a^3 \propto T^2$ for *all* eccentricities, there is none to be found in the *Principia*. Hence, the claim is unsupported both mathematically and empirically.

Eliminating the influence of the minor axis and eccentricity is equivalent to converting the two-dimensional motion of an ellipse into linear motion. Therefore, this process will now be examined in the preliminary propositions of Book I, Section 7: “The rectilinear ascent and descent of bodies.”

3.1. Proposition XXXII

In this proposition, the rectilinear descent of a body is compared with motion on an ellipse, with both sharing the width of the major axis a :

Prop. XXXII. Prob. 24. Given a centripetal force inversely proportional to the square of the distance of places from its center, to determine the spaces which a body in falling straight down describes in given times.

The relevant figure for Case I (ellipse) is reproduced below from [25]:

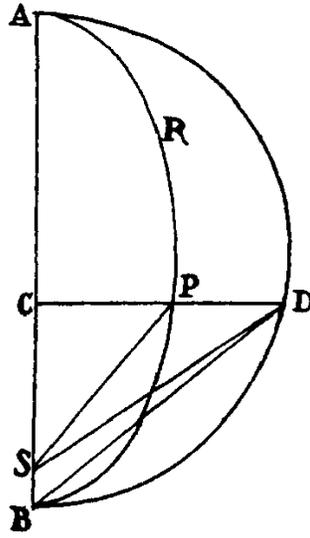


Fig. 2. ADB is a semicircle, ARP forms an elliptic arc, with particle at P. S is the focus, and $AB = 2a$.

In the description, Newton does not begin with what one might expect, viz. with Proposition XV or XVI, such that he can prove in one step:

$$\frac{T_{circle}^2}{T_{ellipse}^2} = \frac{a_{circle}^3}{a_{ellipse}^3} = \frac{a^3}{a^3} = 1 \Rightarrow T_{circle} = T_{ellipse} = T_{line\ ABA} \quad (5)$$

Line ABA simply means that the circular motion is converted to a back-and-forth motion between extremes. From here, it only needs to be pointed out that since time periods are equal, time taken for a particle to traverse the circular arc AD *could perhaps* be the same as that for the linear segment AC , subject to other considerations like the area law (Kepler's second law). Even if Proposition XV is compromised as discussed earlier, this would be the logical place to start. However, Newton appears to *begin* with Kepler's second law, for *different* orbits, by comparing areas proportional to times as (*italics mark the equations that follow*):

... area ASD will be proportional to area ASP and thus also to the time. (6a) Keeping the axis AB fixed, continually diminish the width of the ellipse, and area ASD will always remain proportional to the time. Diminish that width indefinitely; and, the orbit APB now coming to coincide with the axis AB, and the focus S with the terminus B of the axis, the body will descend in the straight line AC (6b), and the area ABD will become proportional to the time (6c). Therefore the space AC will be given...

This is quite unexpected, since the constant in the area law is defined for one particular orbit, and there is no guarantee that A/T is the same constant for both orbits. Nevertheless, Newton's description can be written out as a series of elementary equations. From the quote above, (and the second law):

$$\frac{\text{Area } APS}{\text{Area } ADS} = \frac{T_{APS}}{T_{ADS}} \quad (6a)$$

Taking the limits as the ellipse shrinks to a line, and point S moves to point B:

$$\lim_{P \rightarrow C} APS = \text{line } AB ; \lim_{P \rightarrow C} ADS = ADB \quad (6b)$$

Substituting (6b) into (6a), one gets:

$$\lim_{P \rightarrow C} \frac{\text{Area } APS}{\text{Area } ADS} = \frac{\text{Area } (\text{line } AB)}{\text{Area } ADB} = \frac{T_{AB}}{T_{ADB}} = 0 \quad (6c)$$

Since the area shrinks to zero, and the time period for the circular orbit is a nonzero constant, this implies that the time taken to cross AB is zero, i.e. instantaneous. On the other hand, if one assumes from equation (5) that $T_{AB} = T_{ADB}$, that would imply that the line AB has a finite area.

This proof is untenable in this form. Yet, the same procedure is followed even in Case 2 (parabola) and Case 3 (hyperbola) of the same proposition. In these two cases one cannot even take the aid of equation (5), since the equation applies only for closed orbits. It is hard to extricate any meaning from this proposition, and the only possibility that remains is a (flawed) proof for ellipses using equation (5).

In reality, there are two major problems with the method of comparing an ellipse to a line. Firstly, equation (5) is assumed, and not proved, making it impossible to compare movements between ellipses of different eccentricities. Secondly, even if equation (5) was true, there is no prior reason to suppose that the time taken for arc AD is the same as the time taken for segment AC, because the relation $T_{\text{ellipse}} = T_{\text{line } ABA}$ holds *only for total time periods*. This does not tell us anything about velocity variation across the orbit, hence there is no way of determining if the orbiting particle remains on the line CD as the orbit is shrunk. In fact, compared to the case of the circle, one would expect that in an elongated orbit the particle should swing past the lower portion (close to S, say arc PB) much faster than the upper portion (say arc AP). In a circle, the velocity is a constant throughout, and its projection on line AB is such that the velocity slows

both in the upper and the lower ends. Hence, the assumption that by shrinking the orbit, the particle will traverse the arc to come to a point on the line CD is once more untenable, and the limiting process in Equations 6b and 6c becomes invalid. P need *not* go to C as the orbit is shrunk.

3.2. Propositions XXXIII-XXXVII

Proposition XXXIII attempts to establish a relationship between the velocity gained by the particle when falling from A to C, and the velocity of a particle passing through C while circling B. It is claimed that:

... I say that the velocity of a falling body at any place C is to the velocity of a body describing a circle with center B and radius BC as the square root of the ratio of AC (the distance of the body from the further vertex A of the circle or rectangular hyperbola) to $\frac{1}{2}$ AB (the principal semidiameter of the figure)

In this proof, once more, the limit is taken this way:

Now let the width CP of the figure RPB be diminished indefinitely, in such a way that point P comes to coincide with point C...

This is identical to the limiting process in Section 3.1, and is hence susceptible to the same objections: There is no reason why the dynamics should restrict the particle P to the line CD as the orbit is shrunk. Just as S migrates to B as eccentricity goes to 1, the point reached at a particular time can in turn vary as well. Since this placement of P is crucial to the method of proportions that is used in the rest of the proof, without a clear reason for the path CD the entire proof is compromised.

Proposition XXXIV repeats the process of XXXIII with a parabola, hence does not require further comments. In Proposition XXXV, on the other hand, a new feature enters the discussion, even before entering into the details of the proof. The statement of the proposition and the relevant diagram for circular motion (case 1) are reproduced below.

Prop. XXXV, Prob.11: Making the same suppositions, I say that the area of the figure DES described by the indefinite radius SD is equal to the area that a body revolving uniformly in orbit about the center S can describe in the same time by a radius equal to half of the latus rectum of the figure DES.

Note: for a circle, semi-latus rectum equals the radius.

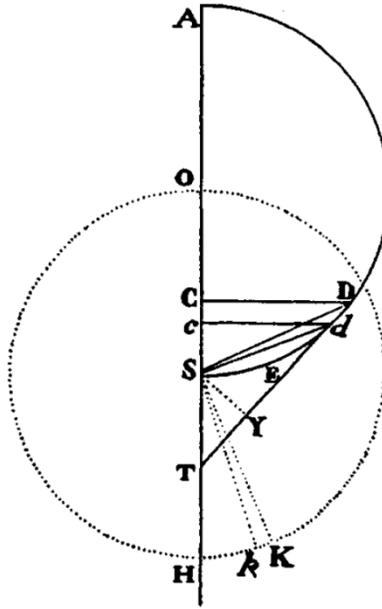


Fig. 3: A particle moving in the path Cc along AS, compared to a particle revolving around the circle centered at S with radius = semi-latus rectum of circle centered at O = radius of circle centered at O.

The primary issue in this proposition has to do with the center S. If the particle is moving along the squashed ellipse/line AS, then with eccentricity equal to 1, the center of force can be S. However, when comparing with the area covered in the circle centered at O, the eccentricity is zero and *the point S also must be situated at O*. One cannot have both: either the point of central force is at S as shown, and the particle traverses a line, or the point of central force is moved to O and the particle's orbit is a circle. In terms of actual dynamics, moving the focus from O to S would mean altering the velocity functions along the orbit as well as changing the orbital shape. *One cannot compare the areas traversed in a circular orbit by setting the focus at the corner of the orbit, at S*. In that case, the circle is no longer a circle, but a line. This would amount to conflating the two cases: linear and circular. Further on, the proof of this proposition utilizes Prop. XXXIII once more, leading to the same issue just discussed, regarding the line CD.

Propositions XXXVI and XXXVII further develop the Propositions XXXIII-XXXV, hence adding nothing new to the dynamics. The error of assuming Kepler's Harmonic Law with only the semi-major axis, as well as the error of conflating a circle into a line without accounting for a change of dynamics, both affect the entire series of Propositions XXXII-XXXVII. It is therefore not possible to convert the Harmonic Law for ellipses into laws for rectilinear ascent and descent, in the manner of the *Principia*.

4. Summary and Conclusions

Since all of Newtonian celestial mechanics hinges on a proper understanding of Kepler's Third Law, or Harmonic Law, the wording of the law has been studied in its entirety in this paper. It has been shown that the form of Kepler's Harmonic Law that is used in the literature, with reference to the semi-major axis alone, is primarily Newtonian – and ignores the constraint introduced by Kepler that the Law works in the way he had presented it only for small eccentricities. Omission of this portion of the law results in a lack of clarity of what is meant by the “mean distance”. An alternate form of mean distance, in keeping with the comparisons given by Kepler in the *Astronomia Nova*, has been presented that agrees quite well with the data Kepler had in hand. By an analysis of the dynamics that result by ignoring the constraint on eccentricity, as it is done in the *Principia*, it is shown that there are several conceptual and mathematical inconsistencies in the proofs for Propositions XXXII-XXXVII. The implicit application of the Newtonian version of Kepler's Harmonic Law in order to make it suitable for rectilinear ascents and descents is shown to be fundamentally flawed.

By combining the results of this paper with the previous [17], it can be seen that many of the central propositions of the *Principia* viz. Propositions X-XVI, XXXII-XXXVIII, suffer from a distortion of meaning, and thereby introduce errors into the application of a force law such as the inverse-square law. This error has been pointed out numerically in the case of Mercury in Section 2, coincidentally the same planet with respect to which the Newtonian approach failed to provide adequate results for the perihelion precession of Mercury. It is quite possible, speculating for the moment, that the discrepancy arises due to the flaw in the inverse-square law, and that it could perhaps be corrected by changing it back to the original form of Kepler's Harmonic Law, and deriving the interrelationships afresh.

It is also interesting to find that there have been some discrepancies found while applying Kepler's Law with only the semi-major axis to the making of nautical almanac [26], while another author [27] has found dynamic problems with the Law as it is applied. This clearly indicates that there is an alternative path to correcting Newton's approach to celestial laws, albeit a path that is more obscure than the rest.

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