# The Quaternion Struggle

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Mathematics was in a bit of a flurry in the beginning of the $19^{th}$ century. Non-Euclidean geometries began to generate interest, and the old rules that had stood for nearly two thousand years were brought into question. The complex number $i$ ($=\sqrt{-1}$) had entered the field and upset the applecart of the mathematicians. It looked like a rotation, but mathematicians were unsure what to attribute it to, while physicists were yet to get wind of it. In the midst of all this, William Hamilton of Dublin, who was aware of the controversy over the complex numbers, set out to find a consistent algebra for these numbers. He realized that this algebra related to the physical concept of time, saying that this "Algebra… viewed not merely as Art or Language, but as the Science of pure Time." If one complex number generates a rotation, he figured, two complex numbers should cover all of 3D space. But instead he found in 1843 that it needed *three* complex numbers: i, j and k! This then maps onto the three rotational axes, and along with the real number, makes up the *Quaternion*, with i, j, and k following these rules:

$$i^2 = j^2 = k^2 = ijk = -1 \quad ij = k,\ ij = -ji$$

$$q = \overbrace{w}^{\text{scalar}} + \overbrace{\mathbf{i}x + \mathbf{j}y + \mathbf{k}z}^{\text{vector}}$$

Hamilton first introduced the terms "scalar" and "vector." He also introduced non-commutativity i.e. the order of operation mattered. This is sensible in terms of rotation, as rotating around x-axis and then along y-axis is different from rotating first around y and then along x. To those who had no idea what *one* complex number meant, let alone three, this looked scary. But they proved quite useful, and Maxwell incorporated them in the famous equations of electromagnetism. This brought them into the domain of physicists, and caught the attention of J. W. Gibbs and O. Heaviside. Both of them, independently, tackled quaternions in Maxwell's works and decided to remove the complex nature of the numbers. Physics of the time had no rotations to map complex numbers to, and as a result, the physicists preferred the linear "real number" version. Heaviside complained: "how can the square of a vector be negative?" So they dropped the complex numbers and forced the vector part of the quaternion into modern Vector Analysis or Vector Algebra, using rules like *cross product* (e.g. right hand rule in electromagnetism.) The scalar part was kept aside, with rules relating to the *dot product*. The quaternion was broken into two convenient pieces: scalars and vectors.

Well, Hamilton's supporters were not going to accept this dismemberment without a fight, and their fight (see *A History of Vector Analysis* by M Crowe) involved eight scientific journals, twelve scientists, and roughly 36 publications between 1890 and 1894. After this, with the increased utility of the vector algebra, practical concerns won the day and quaternions were pushed out of the mainstream. Vector algebra that is still taught today got entrenched into the textbooks.

However, an idea whose time had come could not simply be squashed out of existence simply for convenience's sake. After a couple of decades, the notion of quaternion would again poke out in two different streams. One stream picked up the complex number again and incorporated it into a 4D space-time. This is what we now know as *Special Relativity*. Another stream picked up the non-commutativity as well, giving rise to *Quantum Mechanics*. Paul Dirac, one of the pioneers of this subject, was fascinated by Hamilton's work and even introduced the Hamiltonian equation into quantum mechanics. Quaternions was resurrected again, as were complex numbers, but without a clear connection to their history. All the troubles in understanding quantum mechanics to this day stem from the properties of the complex number and non-commutativity of quaternions, the same thing Hamilton was tackling two centuries ago. Both mathematicians and physicists have been at a loss to explain how physical quantities can be "imaginary", where the rotation called "spin" comes from, and how the order of physical measurement matters. And since imaginary numbers cannot be directly represented on the real line, non-locality was introduced into physics, which was another hard pill to swallow. Understanding the quaternion as an expression of rotation hence not only clears up these problems, but clears the way after nearly two centuries of being lost in the woods.