Celestial Dynamics and Rotational Forces In Circular and Elliptical Motions

Abstract: The understanding of circular motion as being conditioned by a central force coupled to a tangential velocity is re-examined, by analyzing the origins of its derivation, and revising it in the light of rotational kinematics. It is shown that one cannot stop the analysis at a force directed to the center, but has to continue it to include an infinite series of higher order rotational forces in four perpendicular directions. The verification of this in terrestrial dynamics, as well as the consequences of its application in celestial dynamics is presented. It is shown that Newton's Moon Test and inverse-square law, even with the corrections of General Relativity, do not support circular and elliptical motion and lead to an erroneous expression – a problem that has been noticed and partially remedied by other independent researchers.

Keywords: centripetal force, inverse-square law, higher derivatives, jerk, acceleration

1. Introduction

The question of the relation between linear motion and circular motion has a long history. According to Aristotelian understanding, circular motion is primary, and linear motion is secondary, since a circular motion can continue indefinitely and uniformly in a perceptible manner whereas the linear motion has to reverse direction in order to remain a real and perceptible motion (Aristotle, 2008). This priority of circular over linear motion remained for a long time, including at the time of Galileo, who actually considered uniform straight line motion as being secondary to uniform circular motion (Galilei, 1632) :

> … if all integrable bodies in the world are by nature movable, it is impossible that their motions should be straight, or anything else but circular…

It was Descartes who challenged this idea, and held that linear motion was primary (Descartes, 1644):

> 39. The second law of nature: that every motion of itself is rectilinear; and hence what is moved circularly tends always to recede from the center of the circle it describes.

In other words, according to Descartes if an object is in circular motion, it has to be restrained by a force towards the center, since its tendency is to *recede* from the center. In this view, circular motion hence requires a pull, such as a sling pulling on a stone before releasing, and is therefore secondary. This is the image reproduced from his work:

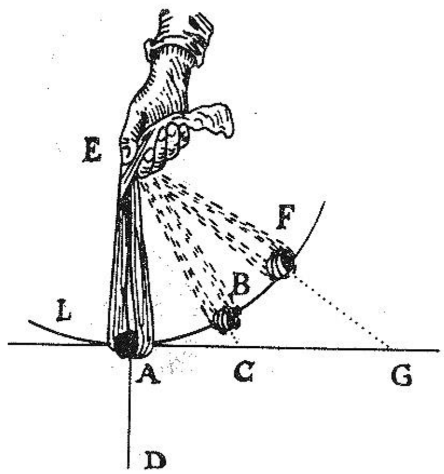

Fig.1: Descartes' image of the slingshot, where the stone would have pursued *ACG* if not held back to follow *ABF*.

The problem of finding this pulling force in a circular motion was first solved in an excellent treatment by Christiaan Huygens in his works *Horologium Oscillatorium* (1658; second edition 1673) and *De Vi Centrifuga* (1703). It was in these works that the following famous relation was first derived (Huygens, 1703):

$$F \propto \frac{v^2}{R} \tag{1}$$

His derivation was extended by Hooke, Wren and Halley to derive the inverse square law of forces for planetary orbits by utilizing Kepler's Third Law (see (Newton & Henry, Circular Motion, 2000)). Newton continued the derivation in the same fashion, and established the law for the force to the center diminishing as an inverse square of the distance. Hence, according to him:

$$F \propto \frac{1}{r^2} \tag{2}$$

But there is one issue that is not addressed: a force or acceleration may be *necessary*, but is it *sufficient* to produce circular motion? If yes, then it can be asserted with full confidence that an inverse-square force is sufficient to generate a circular or elliptical orbit. If not, then there are revisions required.

2. Higher Order "Forces"

Any object in uniform circular motion has the property that its radius and its velocity are mutually perpendicular, have constant magnitude, and constant change in direction. This change is uniform with respect to time. For example, assume an object *P* rotating around a center *C* with a radius $\vec{R}$ and velocity $\vec{v}$. Let the angular velocity, of magnitude v/R, be denoted by ω – which is a constant.

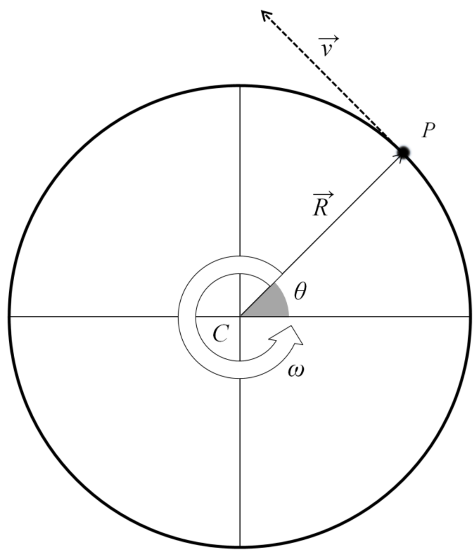

Fig. 2: Uniform Circular Motion of an object *P* around the center *C*, with $\vec{v} \perp \vec{R}$.

Following this, one is not satisfied with the uniform change of position, or the uniform change of velocity, but seeks the cause of this change of velocity in an *acceleration*. This is the acceleration of the particle *P* towards *C*, which is part of the well-known central force. This is where every text on circular motion ceases discussion, since the derivation of the force laws is then given more importance. *By extending the logic, it is necessary to ask the reason for the rotary movement of the acceleration vector as well*. What makes the acceleration rotate, in the same way that the acceleration made the velocity rotate? But it is *assumed* that only the velocity $\vec{v}$ and the acceleration $\vec{a}$ are sufficient to fully define circular motion. However, with circular motion, obtaining the acceleration vector is simply the *first* step after obtaining the velocity vector, as the acceleration also rotates constantly at the *same* angular velocity ω. And so does the derivative of acceleration (called jerk), and 2nd derivative of acceleration (snap) and all further rates of change until infinity. In other words, on expressing the rates of change of the radius vector starting with the velocity $\vec{v}$, one obtains the following series of successive rates of change:

$$\vec{v}, \vec{a}, \frac{d\vec{a}}{dt}, \frac{d^2\vec{a}}{dt^2}, \frac{d^3\vec{a}}{dt^3}, \frac{d^4\vec{a}}{dt^4}, \frac{d^5\vec{a}}{dt^5} \dots \frac{d^\infty\vec{a}}{dt^\infty} \tag{3}$$

Whose magnitudes are denoted by:

$$v, a, \dot{a}, \ddot{a}, \dddot{a}, a^{(4)}, a^{(5)} \dots a^{\infty} \tag{4}$$

The "dot notation" is used for the first three rates of change of acceleration, and the superscript *(n)* shows the n^{th} rate of change of acceleration for the remaining. Each of these terms can be derived as follows, assuming unit vectors in the radial (along radius $\vec{R}$) and tangential (along velocity $\vec{v}$) directions to be $\hat{\mathrm{r}}$ and $\hat{\theta}$ respectively.

$$\vec{R} = R\,\hat{\mathrm{r}}$$

$$\vec{v} = v\,\hat{\theta} = R\omega\,\hat{\theta}$$

$$\vec{a} = -a\,\hat{\mathrm{r}} = -R\omega^2\,\hat{\mathrm{r}}$$

$$\frac{d\vec{a}}{dt} = -\dot{a}\,\hat{\theta} = -R\omega^3\,\hat{\theta}$$

$$\frac{d^2\vec{a}}{dt^2} = \ddot{a}\,\hat{\mathrm{r}} = R\omega^4\,\hat{\mathrm{r}} \tag{5}$$

The magnitudes of these functions with reference to expression (4) are given by:

$$R\omega, R\omega^2, R\omega^3, R\omega^4, R\omega^5, R\omega^6, R\omega^7 \ldots R\omega^\infty \tag{6}$$

And so on, each time getting multiplied by ω and changing direction by 90º. Hence, given a finite R and ω, all the different rates are easily derived. The variation of direction, with respect to what is usually assumed, can be shown as in Fig. 3.

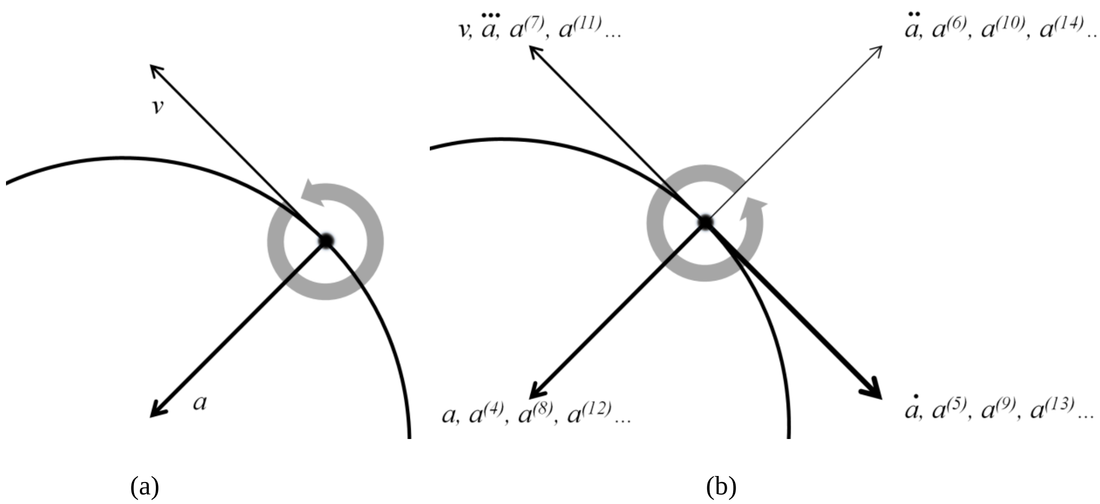

Fig. 3: Directions of successive rates of change, in the radial and the tangential directions, (a) as usually assumed and (b) as they exist. The directions rotate counterclockwise by 90º.

It hence turns out that circular motion is quite a complicated system, with an infinity of derivatives all existing simultaneously. This is very different from the situation usually assumed – that with circular motion one simply has accelerated motion. In reality, one has an accelerated, jerky, snappy, etc. motion all at once. Since mass is combined with velocity to give momentum p, and with acceleration to give force F, one therefore has an infinite series of "higher-order forces" along with them:

$$mv, ma, m\dot{a}, m\ddot{a}, m\dddot{a}, ma^{(4)}, ma^{(5)} \ldots ma^\infty \tag{7}$$

$$p, F, F', F'', F''', F^{(4)}, F^{(5)} \ldots F^\infty \tag{8}$$

Hence from simple geometric reasons, any circular motion of a particle of mass m has to ***necessarily include infinite number of higher order forces, in order to remain circular.*** More importantly, they consist not only of forces directed towards the center of the circle, but also those directed away from the center, and tangential to the movement of the particle. ***Therefore, no circular motion, from a slingshot to a planetary movement, can be derived from centric forces alone***.

It is convenient to group together all the forces in different categories based on the directions shown in Fig. 3:

a. Towards the Center – Centripetal Forces: $F, F^{(4)}, F^{(8)}, F^{(12)}\ldots$

b. Opposite to velocity – Retarding Forces: $F', F^{(5)}, F^{(9)}, F^{(13)}\ldots$

c. Away from Center – Centrifugal Forces: F'', $F^{(6)}$, $F^{(10)}$, $F^{(14)}$…

d. Direction of velocity – Quickening Forces: F''', $F^{(7)}$, $F^{(11)}$, $F^{(15)}$…

It may be objected that the values of the higher order forces are too small in most practical cases. However, that is not conceptually relevant to the discussion, since it is only true if $\omega << 1$, which depends on the *units* being chosen for the angle and time. Even if the actual numerical values were small in a particular set of units, the smallest values have a finite effect in the final function, especially when applied to planetary dynamics. All the derivatives are *mathematically* and *physically* necessary to retain the form of the circle.

Hence, attributing circular motion to a central force *alone* is erroneous. There is currently no name for the entire complex of movements for circular motion along with higher order forces, hence it is possible to use the term "rotational forces" for the net resultant that retains uniform circular motion. In other words, rotational forces are necessary for circular motion to exist.

3. Implications in Terrestrial Physics

The natural question that follows is: If all the higher derivatives are an intrinsic part of circular motion, why aren't they better known?

It turns out that they are indeed well known only in areas of practical application of circular motions (Sandin, 1990), such as designing train tracks (Royal-Dawson, 1932), road pavements (Hugo & Martin, 2004), roller coasters (Schützmannsky, 2008), and circular machine parts (Faires, 1965). One particular researcher refers to an interesting phenomenon (Theron, 1995):

> It is shown that an elastic wheel rolling down a circular track onto a horizontal pavement will, under certain conditions, bounce off the track due to the jerk it experiences... The wheel has a surprisingly large response to the sudden change in acceleration at the end of the circle, another example of the effect of infinite jerk.

Other authors frankly declare (Eager, Pendrill, & Reistad, 2016):

> Jerk is a common everyday experience, but rarely mentioned in the teaching of mechanics.

Hence, the requirement of higher derivatives of circular motion is not mathematical alone, but has a clear physical application. It necessarily means that to generate circular motion, one has to generate not only a central acceleration, but also all the rotational forces. The track in case of a train, the friction in case of a road, the hand in case of slingshot or the axle in case of a wheel hence bear the combined effect of the acceleration and higher derivatives, acting both toward and away from the center, along and opposite to the velocity.

A note is needed about the simple experiment done in all physics labs across the world to reproduce Huygens' relation $F = mg = Mv^2/r$. The diagrams usually show something like this:

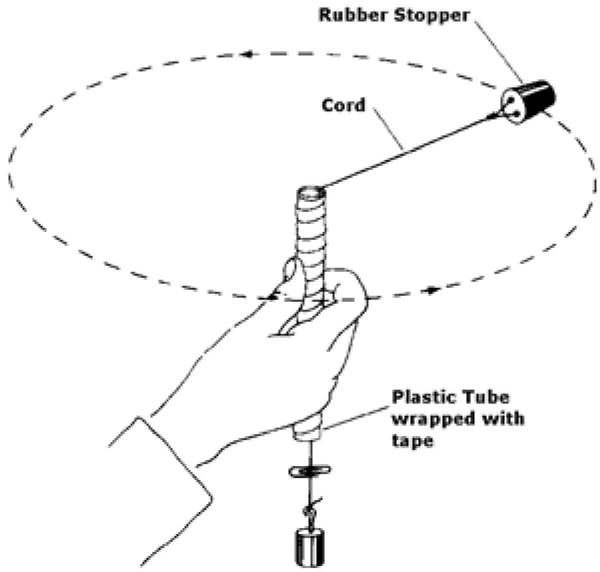

Fig. 4: Centripetal force measured by hanging weights (*m*) to a rotating object (*M*).
Sometimes the hand is replaced by the movement of a motorized axis.

Does this conclusively show that rotational motion involves only acceleration, since it is balanced by acceleration due to gravity? On the contrary, it merely shows that gravity is able to balance *one* of the components of rotational motion, while the axis (hand or motor) generates all the other components. It is the component one *chooses* to measure that one ends up measuring in the experiment. If other derivatives are generated, by hanging variable weights and so on, then the values of other rotational forces can also be demonstrated – something not pursued in the teaching field.

Hence, the existence of rotational forces – some *source* of the entire rotation – is essential for a full description of circular motion even under normal terrestrial conditions. Any error in their application for terrestrial phenomena prevents them from predicting celestial movements.

4. Implications in Celestial Dynamics

There is hence a clear need to determine the physical implications of circular orbital motion, for which it is necessary that all of the rotational forces be present. There are four major repercussions for celestial dynamics:

1. If for one of the movements (central acceleration) a force is postulated to exist (gravity), then it is equally important to *postulate infinite higher order forces* in the other three directions.

2. The number of centripetal forces (F, $F^{(4)}$, $F^{(8)}$, $F^{(12)}$… etc.) are *equal* to the number of centrifugal forces (F'', $F^{(6)}$, $F^{(10)}$, $F^{(14)}$… etc.)

3. The number of rotational forces that act in the direction of the velocity (F''', $F^{(7)}$, $F^{(11)}$, $F^{(15)}$… etc.) are *equal* to the number of rotational forces acting opposite to that (F', $F^{(5)}$, $F^{(9)}$, $F^{(13)}$… etc.)

4. All the rotational forces have an angular velocity, and rotate.

In celestial dynamics, usually the focus is only on the tangential velocity '*v*' and central acceleration '*a*' for circular motion.

Velocity '*v*': The reason for velocity in the planetary system is generally attributed to some variation of the '*nebular hypothesis*,' where the entirety of rotational forces is supposed to have been present in a rotating primordial nebula (first proposed by Kant (Kant, 1755) and Laplace (Laplace P.-S. , 1796).) This theory has remained more or less in the same form for two centuries – ever since it was proposed. But unless it is shown how any physical mechanism can possibly generate an *infinite series of tangential forces out of nothing*, the whole idea deteriorates into empty speculation.

Acceleration 'a': The entire burden of accounting for circular motion is currently on gravitational acceleration. The variation is primarily in the radial direction, unlike the tangential for the velocity. Since there are equal number of forces pointing towards and away from the center, even if all the collective forces directed towards the center the "generic gravity" then there has to be a generic force pointing away from the center. In other words, if *gravity* – a pull – exists, then so should *levity* – a *push*. By focusing on only the acceleration and ignoring higher orders, the entire symmetry of the circular motion is destroyed without justification.

Even works that derive the entirety of orbital motions of planets to satellites make no mention of the fact that multiple higher order forces – which are the counterparts to gravity – are demanded by the physical situation (See (Tan A. , Chapter 3, Section 3.8, 2008)). The most common opinion is that the force of acceleration is directed towards the center, and it is only in a rotating frame of reference (such as a car taking a turn) that one "feels" a fictitious outward force. But as seen in the discussion in the previous section, the analysis did not require any change of references, as the rotational forces show up with respect to the stationary point *C*. Fictitious forces are not relevant there.

Since acceleration is the only quantity being treated, all variations for acceleration were done by modifying the force law itself, by which we get the different aspects of modern gravitation theory (k_i denotes a constant):

$$F = -\frac{k_1}{r^2} \qquad \Rightarrow \qquad \text{Newton's Law of Gravitation}$$

$$F = -\frac{k_1}{r^2} + \frac{k_2}{r^3} \qquad \Rightarrow \qquad \text{Newtonian Perturbation for precessing ellipses}$$

$$F = -\frac{k_1}{r^2} - \frac{k_3}{r^4} \qquad \Rightarrow \qquad \text{Einstein's General Theory of Relativity (Eddington, 1963)}$$

It is clear that the consequence of ignoring the complete set of rotational forces has been to attempt approximating it by modifying the existing force law solely with acceleration.

The argument of the Newtonian approach hinges on two important criteria:

1. The Moon Test: Where Newton indicated that the earthly gravity and moon's centripetal force are related.

2. Motion in an Ellipse: Where the inverse square law is said to hold true for motion in an ellipse.

Both of these criteria will be examined in Sections 5 and 6.

5. The Moon Test

In light of this, consider the moon test, as shown in Proposition 4 Book III of Newton's *Principia*, where it is declared (Newton I. , Book III, 1999):

> Prop. IV Theorem 4. The moon gravitates toward the earth and by the force of gravity is always drawn back from rectilinear motion and kept in its orbit.

It is clear that gravity or rectilinear acceleration is the main comparison and there is no mention of the rotational forces. If all forces were included, the moon has to be drawn back, drawn forward, and drawn away all at once to keep it in its orbit. Thereafter, the supposition follows:

> If now the moon is imagined to be deprived of all its motion and to be let fall so that it will descend to the earth with all that force urging it by which (by prop. 3, corol.) it is [normally] kept in its orbit, then in the space of one minute, it will by falling describe 15 $^{1}/_{12}$ Paris feet. This is determined by a calculation carried out either by using prop. 36 of book 1 or (which comes to the same thing) by using corol. 9 to prop. 4 of book 1.

Here Newton uses a relation (Prop 4. Corr. 9) that is true only in the infinitesimal limit, to a process that takes finite time of 1 minute (see (Denison, 1846).) Thereafter, noting that the ratio of distance to moon (R_m) and the earth's radius (R_e) is ≈ 60, Newton showed the numerical equality:

$$\frac{v_m^2}{R_m} = \frac{g}{60^2} \tag{9}$$

Here g is the acceleration due to gravity on the earth's surface, and v_m the orbital velocity of the moon. What is meant in Proposition IV by the phrase "deprived of all its motion"? To assume that the moon is deprived of its angular velocity, one must also assume that the acceleration disappears, as it is part of the same complex of rotational forces. When ω is zero, so are all the higher derivatives, from expression (6). This relation (9) appears to hold good only with the assumption that it is possible to convert all of the rotational forces into a rectilinear accelerative motion, *an assumption for which no evidence is presented*.

Even if the acceleration due to gravity is varying as the inverse square with the distance, this could well be a geometric effect of the curvature of the earth and have nothing to do with the cause of the Moon's motion. *One* numerical coincidence is no guarantee of the truth of a physical law, and this fact becomes especially true in the solar system where several coincidences, such as the equal angular sizes of the Sun and Moon from earth and Bode's Law, are prevalent. Since there is no mathematical or physical reason to ignore the presence of the rotational forces, any theory that is built on such an assumption remains fundamentally mistaken.

6. Elliptic Motion

Newton indicated that for an object to move in a conic section, the acceleration has to be the inverse-square of the distance from the focus. Motion in an ellipse, just like the circle, also involves higher derivatives of acceleration. Therefore, assuming the equation for the ellipse, and also Kepler's second law (areal velocity is constant), one can derive both the acceleration and its derivative – the "jerk". This will be called the *Geometrical Method*. If the inverse square law is true, the same value of jerk must be obtained by using the inverse square expression for acceleration. This will be called the *Inverse-Square Method*. These two derivations will be compared here.

6.1 Geometric Method

In this, the third derivative of the radius vector is obtained by successive differentiation with respect to time. The only input to the process is the equation of an ellipse in polar coordinates:

$$\frac{l}{r} = 1 - e\cos\theta \tag{10}$$

Here, l is the semi-latus rectum of the ellipse, and e (<1) is its eccentricity. The other input is the area law of Kepler (second law) which is expressed as:

$$r^2\omega = r^2\dot{\theta} = h = constant \tag{11}$$

Also, for unit vectors in polar coordinates:

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta} \quad \text{and} \quad \frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r} \tag{12}$$

The position vector is simply:

$$\vec{r} = r\hat{r} \tag{13}$$

First derivative:

$$\frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \tag{14}$$

Second derivative:

$$\frac{d^2\vec{r}}{dt^2} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} \tag{15}$$

The tangential section is generally set to zero by using equation (11) at this point. But for the third derivative, this operation is not valid (as is done, for instance, in (Tan A. , The Jerk Vector in Planetary Motion, 1992)), as it is possible in rotating coordinate systems for the tangential component's derivative to exist even if the component itself does not, since the unit vectors themselves are rotating. Therefore the third derivative is taken first:

$$\frac{d^3\vec{r}}{dt^3} = \left(\dddot{r} - 3\dot{r}\dot{\theta}^2 - 3r\dot{\theta}\ddot{\theta}\right)\hat{r} + (3\ddot{r}\dot{\theta} + 3\dot{r}\ddot{\theta} + r\dddot{\theta} - r\dot{\theta}^3)\hat{\theta} \tag{16}$$

This is the "jerk" vector j. It is important to focus on the radial part, since that is the part that affects the central force relations. Utilizing equations (10) and (11) *here*, and substituting in the expressions in equation (16), one gets:

$$\frac{d^3\vec{r}}{dt^3} = \vec{J}_1 = -\frac{eh^3}{l}\left(\frac{2\sin\theta}{r^4} + \frac{e\sin 2\theta}{lr^3}\right)\hat{r} - \left(\frac{3h^2 e cos\theta}{lr^2} + \frac{h^3}{r^5}\right)\hat{\theta} \tag{17}$$

This is the expression to be used for comparison with the Inverse-Square method.

6.2 Inverse-Square Method

For this, the starting point is simply the expression used for Newton's inverse-square law:

$$\frac{d^2\vec{r}}{dt^2} = -\frac{h^2}{l}\hat{r} \tag{18}$$

Differentiating this:

$$\frac{d^3\vec{r}}{dt^3} = \vec{J}_2 = -\frac{2eh^3}{l^2}\frac{\sin\theta}{r^3}\hat{r} - \frac{h^3}{lr^2}\hat{\theta} \tag{19}$$

With some simplification of the radial part of equation (17), it matches the radial part of (19):

$$\left.\frac{d^3\vec{r}}{dt^3}\right|_r = \vec{J}_{2r} = -\frac{2eh^3}{l^2}\frac{\sin\theta}{r^3}\hat{r} = -\frac{eh^3}{l}\left(\frac{2\sin\theta}{r^4} + \frac{e\sin 2\theta}{lr^3}\right)\hat{r} = \vec{J}_{1r} \tag{20}$$

However, it is quite obvious from comparing (17) and (19) that the two tangential jerks are not equal:

$$\frac{3h^2 e cos\theta}{lr^2} + \frac{h^3}{r^5} \neq \frac{h^3}{lr^2} \rightarrow \vec{J}_{1\theta} \neq \vec{J}_{2\theta} \tag{21}$$

In other words, the inverse square law is *not consistent* with elliptical motion geometrically even for the first derivative beyond acceleration. It naturally cannot hold for all the higher derivatives. Hence, this law is tailor-made for rectilinear acceleration alone, and does not hold for closed orbits like the ellipse. In other words, even if the inverse square law holds for the acceleration due to gravity, it does not hold for the forces necessary for a body to move in an ellipse. This means that the basis of defining an inverse-square law, and therefore, the application of such a law to planetary bodies becomes invalid.

7. Alternatives

From the preceding sections, it can be deduced that:

a. Circular and Elliptical orbits require the presence of an infinite series of rotational forces to maintain them.

b. The Newtonian inverse-square law fails to account for these forces, and chooses *one* force without justification.

It is therefore instructive to compare rectilinear and rotational motion in the light of this knowledge, and conclude that the two are *incommensurable*. Since it requires infinite series of radial and tangential forces to represent circular motion, it follows that one cannot obtain uniform circular motion simply by using a force, as assumed by Descartes, Huygens and Newton. On the other hand, it implies that a different approach is necessary to tackle this process that both *recognizes* the inadequacy of this approach and *supplements* it with a fresh train of thought.

For this, it is necessary to retrace the development historically, to identify who else had suggested an alternative approach. One of the primary objections to Newtonian celestial dynamics came from the philosopher G W F Hegel (1770-1831) who criticized Newton's approach based on his Natural Philosophy, worth reproducing here (Hegel, 2004):

> It is admitted by mathematicians themselves that the Newtonian formulae may be deduced from Kepler's laws. The quite abstract derivation, however, is simply this: In Kepler's third law, the constant is A^3/T^2. If this is put in the form $A.A^2/T^2$, and we call A/T^2 with Newton, universal gravitation, then we have his expression for the action of this so-called gravitation, in the inverse ratio of the square of the distances.
>
> Here, therefore, we believe we have a law which has for its moments: 1. the law of gravitation as the force of attraction; 2. the law of the tangential force. But if we examine the law of planetary revolution we find only

> one law of gravitation; the centrifugal force is something superfluous and thus disappears entirely, although the centripetal force is supposed to be only one of the moments. This shows that the construction of the motion from the two forces is useless. The law of one of the moments—what is attributed to the law of attraction—is not the law of this force only, but reveals itself to be the law of the entire motion, the other moment becoming an empirical coefficient. Nothing more is heard of the centrifugal force.

Hegel here expresses his opposition to the idea of distorting Kepler's Third Law and retaining only the single centripetal force by abandoning the centrifugal force. He expresses philosophically, the same things that have been demonstrated mathematically in this work: it is not possible to eliminate the forces acting away from the center.

This fact has also been mentioned in another context by Steiner (Steiner, Third Scientific Lecture Course, Astronomy Lecture III, Stuttgart, 1921) and extended thus (Steiner, Third Scientific Lecture Course, Lecture X, Stuttgart, 1921):

> … the reality confronting us in this case is in fact like the realm beyond the sphere in its relation to what is within the central point. However we look to the phenomena of the heavens, we must recognise that we cannot study them simply according to the laws of centric forces, but that we must regard them in the light of laws which are related to the laws of centric forces as is the sphere to the radius.
>
> If, then, we would reach an interpretation at all of the celestial phenomena, we must not arrange the calculations in such a way that they are a picture of the kind of calculations used in mechanics in the development of the laws of centric forces; but we must formulate the calculations, and also the geometrical forms involved, so that they relate to mechanics as sphere relates to radius. It will then become apparent … that we need: In the first place, the manner of thinking of mechanics and phoronomy, which has essentially to do with centric forces, and secondly, in addition to this system, another, which has to do with rotating movements, with shearing movements and with deforming movements. Only then, when we apply the meta-mechanical, meta-phoronomical system for the rotating, shearing and deforming movements, just as we now apply the familiar system of mechanics and phoronomy to the centric forces and centric phenomena of movement, only then shall we arrive at an explanation of the celestial phenomena, taking our start from what lies empirically before us.

What has been described above is the exact conceptual analogue of the prior discussions of the types of forces needed to maintain circular motion. The reference to the relation of "sphere to the radius" points to the two different directions of forces: one directed *towards* the center and one directed *away* from the center – to the periphery. This necessity for including at least two forces for rotational equilibrium has also been discussed by Dewey Larson in the galactic context:

> Gravitation is normally visualized as a *force*, but in the case of the isolated galaxies, where no opposing forces are present, it is obviously a *motion*, and since the gravitational motion of each galaxy is directed *inward* toward all other galaxies, this gravitational motion is directly opposed to the motion of the space-time progression, which carries each *galaxy outward* away from all others. (Larson, Ch. 3: The Answer, 1964)
>
> But now we find that there is a second "general force" that has not hitherto been recognized, just the kind of an "antagonist" to gravitation that is necessary to explain all of these otherwise inexplicable phenomena. Just as gravitation moves all units and aggregates of matter inward toward each other, so the progression of the natural reference system with respect to the stationary reference systems in common use moves material units and aggregates, as we see them in the context of a stationary reference system, outward away from each other. The net movement of each object, as observed, is determined by the relative magnitudes of the opposing general motions (forces), together with whatever additional motions may be present. (Larson, Ch. 3: Reference Systems, 1959)

Each of these approaches all point in a direction that has seldom been taken in the astronomical sciences. It is essential to take them up again.

8. Conclusion

The inclusion of forces of levity may appear very similar to the Aristotelian notions of natural motions, but contrary to the way they were expressed in most of the Middle Ages, in this case they are expressed both mathematically and conceptually, and shown to have more validity than is generally assumed. It is only by ignoring some aspects of physical phenomena and retaining others is it possible to sustain a theory of universal gravitation. The fact that an enormous number of calculations have been possible due to this law is no guarantee of its validity – even the epicyclic theories of Ptolemy enabled calculations to be carried out in a similar fashion for centuries through various ingenious devices. The replacement of geometric circular epicycles with algebraic perturbative epicycles using differential equations adds nothing conceptually new to the situation, and continues the same conceptual flaws of its Ptolemaic precursor. A case in point is the calculation of lunar movement by Charles Delaunay during 1860-67 (Linton, 2004):

> However, each step of the process requires a hugely laborious algebraic calculation, and Delaunay's theory – which was taken to eighth order in *m*, *e*, and *i* , sixth order in *e'*, and fourth order in *a/a'* – involved a series expansion of 20 terms and required fifty-seven transformations!

The reality of circular and elliptic motions also show the impossibility of actually calculating the forces needed to sustain the motion – a direct consequence of the fact that π is transcendental and not algebraic – making it impossible to represent it in finite terms. Rather than attempting to reduce circular motion to a series of linear accelerations and higher-order forces, it is preferable for astronomy to deal with circular motion on its own terms, in a descriptive fashion. Circular motion does seem to be irreducible, and if one is not able to conceptually apply current astronomical theory to the moon – the simplest circular motion that we perceive – then it is impossible to do so for the other planets. In terms of actual reality, modern astronomy is seen to be conditionally valid only in the *sub-lunar* sphere, and all application of it to celestial movements is an unjustified extrapolation.

Thus, the notion of circular motion has been found to be central to the arguments against an inverse-square law of gravity. It has been shown that one cannot stop at "centripetal acceleration", but is logically obliged to define an infinite series of acceleration derivatives and higher-order rotational forces in order to account for the motion. This defines four sets of forces towards and away from the center, as well as tangential to it. Assuming an inverse-square force in spite of its inadequacy is shown to lead straightaway to an erroneous value for the "jerk" needed to maintain elliptical motion, thus removing the fundamental basis of applying this force to celestial dynamics. This also removes the conceptual basis for the corrections of General Relativity. Lastly, a series of researchers have pointed out this problem as well as alternative approaches to the problem of celestial rotations. It is suggested that these lines of research must be investigated further in order to provide a surer basis to the theory of astronomy.

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